Introduction

This book is devoted to the description of metamaterials, their origins and physical principles, their electromagnetic and optical properties, as well as to their potential applications. This field has witnessed an immense gain of interest over the past few years, gathering communities as diverse as those from optics, electromagnetics, materials science, mathematics, condensed matter physics, microwave engineering, and many more. The field of metamaterials being therefore potentially extremely vast, we have limited the scope of this book to those composite materials whose structures are substantially smaller, or at the least smaller, than the wavelength of the operating radiation. Such structured materials have been called metamaterials in order to refer to the unusual properties they exhibit, while at the same time being describable as effective media and characterized by a few effective medium parameters insofar as their interaction with electromagnetic radiation is concerned. We also include in this book a chapter on photonic crystals, which work on a very different principle than metamaterials, but which have been closely connected to them and have been shown to exhibit many similar properties.

The metamaterials discussed in this book are designer structures that can result in effective medium parameters unattainable in natural materials, with correspondingly enhanced performance. Much of the novel properties and phenomena of the materials discussed in this book emanate from the possibility that the effective medium parameters (such as the electric permittivity and the magnetic permeability) can become negative. A medium whose dielectric permittivity and magnetic permeability are negative at a given frequency of radiation is called a negative refractive index medium or, equivalently, a left-handed medium, for reasons that will become clear shortly. In this book, we do not, however, discuss another important and powerful manner of attaining extraordinary material properties – that of coherent control whereby atomic and molecular systems are driven into coherence by strong and coherent electromagnetic fields (Scully and Zubairy 1997). Due to the extremely coherent nature of the excitation and response, the quantum mechanical nature of the atoms and molecules is strongly manifested in these cases and the description of the atomic systems relies necessarily on quantum mechanics. In contrast, we remark that since the sizes of the metamaterial structures we are interested in are microscopically large (compared to atomic sizes) and the resonances reasonably broad, it is the classical electromagnetic properties that are apparent. Hence, we ignore the quantum mechanical nature of light.
and matter throughout our discussions.

This chapter offers a general introduction to the topic this book is devoted to, starting with a brief description of the historical development of the subject. We give first a general account of the development of optics, electromagnetism, and the characterization of their effects by effective constitutive parameters. A more specific account of the development of the ideas surrounding metamaterials and negative effective medium parameters follows. We then clarify mathematically our definitions, and discuss the Lorentz model for the dispersion of the dielectric permittivity of a dispersive medium and the basic definitions for the description of negative refractive index media. In the course of this chapter, we hope to set out the basic foundations that will allow the reader to follow the book without much confusion.

1.1 General historical perspective

The study of optical phenomena has accompanied the evolution of mankind from almost its origins. Astronomy, which is often said to be the oldest of all science, has led humans through an incredibly vast journey of discoveries, turning philosophers into scientists and passive observations into active research. During the last couple of centuries, man has achieved an unprecedented understanding and control over light thanks to one fundamental property: light exhibits just the right amount of interaction with matter. This interaction is intense enough compared to that between other particles or matter waves such as neutrons or neutrinos, and yet weak enough compared to the interaction between charged particles such as electrons or protons. The fact that light, or the photon, is one of the fundamental particles of nature and that its propagation velocity sets the ultimate limit on the speed of any signal further underlines the significance of this control.

Despite being one of the oldest topics in science, Optics has remained a very fundamental area of physics and engineering because of the simplicity of its theoretical grounds. It is, for example, formidable to realize that optical properties of many materials can be characterized by a single number called the refractive index, $n$. This number allows one to understand refraction processes and enables the design of lenses and prisms that led to the understanding of colors and dispersion. For a long time, this refractive index was a number that represented the optical density of a medium, a notion reasonably supported by the definition of the refractive index as

$$ n = \frac{c}{v}, \quad (1.1) $$

where $c$ is the speed of light in vacuum and $v$ is the speed of light in the medium.
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Figure 1.1 An adaptation of Ibn Sahl’s original drawing showing refraction at a planar interface. AO is the incident ray from inside the crystal and OB is the refracted ray. Ibn Sahl obtained the reciprocal of the refractive index as \(1/n = \frac{OB}{OC} = \frac{OB}{OD} = \frac{\sin \theta_i}{\sin \theta_t}\) in today’s terminology).

The roots of Optics as a science go as far as the ancient Greek civilization, where Aristotle, upon studying visual perception, recognized the importance of the medium in-between the eye and an object. Another Greek astronomer, Ptolemy, performed several experiments on the effects of refraction on visual perception of objects in the 2nd century AD. Despite these early works, the real credit for the association of a number to the refraction effects of a transparent medium is probably due to Ibn Sahl, an Arabic scholar of Catalanian origin. Ibn Sahl, who lived in Baghdad around 984 AD, wrote a treatise on *Burning Instruments* where he clearly stated a law of refraction for light passing across a plane interface from a material medium into air. This law, completely identical to what we now call the Snell law’s of refraction, defined the refractive index \(n\) in terms of the incident and refracted rays as shown in Fig. 1.1. Ibn Sahl further used this refractive index for a crystal to study the focusing properties of a biconvex lens and several other focusing instruments.

For a long time all optically transparent crystals were mainly characterized by the refractive index. Based on the experimental findings of Willebrord Snellius in 1621, the French philosopher René Descartes (1596–1650) published in his “Dioptrique” the law of refraction in the form we know it today. The refractive index was considered to be a quantification of the resistance offered by a medium to the passage of light. Based on this idea, Fermat enunciated his famous *Principle of Least Action*, which proved invaluable for studies of light propagation in media with spatially varying refractive index. Erasmus Bartholimus had discovered the double refraction in calc-spar in 1669 which led to the realization that there was a polarization associated with light. Malus also discovered polarization, and the rotation of polarization of light

\*The reader is referred to Rashed (1990) for a lucid description of Ibn Sahl’s work.
upon passage through an anisotropic medium in 1808, again with a calc-spar crystal. This was one of the first cases when the optical material could not be characterized by a single number, but the description necessarily depended on the propagation direction and the relative orientation of the crystal.

In parallel to the development of optics, the 19th century also witnessed the emergence of the theories of electricity and magnetism. A plethora of experimental observations challenged the physicists to look for underlying explanations and gave birth to fundamental laws such as Ampère’s, Gauss, or Faraday’s laws. Yet, electricity, magnetism, and optics were seen as independent fields, ruled by independent laws and yielding independent applications. It took the incredible insight and genius of James Clerk Maxwell (see Fig. 1.2) to first unify the former two, and then all the three fields under a uniquely simple and complete theory. With his work, Maxwell showed that electricity and magnetism are entangled phenomena, inseparable, and self-sustaining, ruled by four simple equations known today as the Maxwell equations. The concepts of dielectric permittivity and magnetic permeability, denoted by the letters \( \varepsilon \) and \( \mu \), respectively, became fundamental for the description of media and their response to electric and magnetic fields, and were called constitutive parameters. Moreover, upon studying the self-sustaining solutions of the electromagnetic field in vacuum, Maxwell discovered electromagnetic waves, effectively revolutionizing the field for a second time with descriptions of frequency, wavelength, and propagation speed, with all their fundamental and technological impacts. Finally, upon calculating the propagation speed of the newly discovered electromagnetic waves, Maxwell realized that it was very close to that of light in vacuum, which led him to bridge the two independent fields by declaring that light is an electromagnetic wave. The independent demonstrations of radio frequency waves and their propagation in vacuum by H. R. Hertz, N. Tesla, J. C. Bose, † and G. Marconi, as well as the theoretical work of Einstein obviating the need for the all permeating “aether,” made quick developments in optics possible by utilizing the Maxwell equations.

The connection between the two fields, optics and electromagnetics, was summarized by the very simple equation (also known as the Maxwell relation)

\[
n^2 = \varepsilon \mu, \tag{1.2}
\]

relating the index of refraction, an optical quantity, to the permittivity and permeability of media, two electromagnetic quantities. It was also then realized that all media could be described by the concepts of permittivity and permeability, whose definitions had to be properly generalized. Hence, absorption of light in materials was described by complex valued \( \varepsilon \) and \( \mu \), whereas many anisotropic crystals (where all directions are not equivalent) were described by second-rank tensors \( \tilde{\varepsilon} \) and \( \tilde{\mu} \), effectively yielding different values in different propagation directions or for different polarization states.

†J. C. Bose is credited with the discovery of millimeter waves.
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Figure 1.2 Two giants of electromagnetism: J. C. Maxwell (left) mathematically unified Electricity, Magnetism, and Optics through his equations. The image is taken from the Wikipedia project, http://www.wikipedia.org. H. A. Lorentz (right) gave a microscopic model for the dispersion of the dielectric permittivity. (Courtesy of C. W. J. Beenakker, from the “Collection Instituut-Lorentz, Leiden University.”)

Although dispersion of the refractive index with frequency was a well-known empirical fact by then, it was Fresnel (of the diffraction fame) who first tried to explain it in terms of the molecular structure of matter. This was also supported by Cauchy who gave the well-known dispersion formula which goes by his name. But it was essentially H. A. Lorentz (see Fig. 1.2, right) who gave a reasonably robust theory of dispersion in terms of the polarization of the basic molecules constituting a material. This Lorentz theory of dispersion (described in Section 1.3.2) has been very successful at describing the variation of the dielectric permittivity with frequency and is used as a workhorse model for describing the dispersion in resonant systems. At frequencies well away from an absorption resonance, the Lorentz theory easily approximates into the Cauchy dispersion formula. In dense media (high pressure gases, liquids, and solids), it had to be corrected for local field effects—effects of other neighboring polarized molecules, which yielded the Lorentz-Lorenz model, akin to the Clausius-Mossotti relations for the electrostatic case (Jackson 1999).

Interestingly, although there was no a priori bound on the values of the constitutive parameters, all known transparent media were described by a refractive index between about 1.2 and 1.9 only at optical frequencies.‡ These bounds were broken for the first time when it was realized that stratified

‡Excluding semiconductors where it could be as large as 4 in the infrared regions.
materials, where layers of transparent materials with different refractive indices are stacked together, could exhibit very different optical properties due to well-controlled interference phenomena of the multiply scattered waves at the interfaces between the different media. The most striking examples of such technology are the quarter wavelength anti-reflection coatings and high reflection thin film coatings. The theory of periodic media was later generalized to higher dimensions, making the layered medium a special case of structures later to be called photonic crystals, where strong modifications of the properties of electromagnetic radiation come from multiple scattering or Bragg scattering within the structure. A drastic example is the realization of structures in which light is not able to propagate at all in any direction in a band of frequencies (bandgap) because of the proper interplay of scattering and destructive interference. Actually the realization of a one-dimensional stop-band structure should be credited to Lord Rayleigh who was probably the first to systematically investigate the wave propagation in layered materials (Rayleigh 1887). Lord Rayleigh had already realized the existence of a stop-band and the fact that a layered medium would cause complete reflection of the incident light for frequencies within this band. For further reading on these topics, the reader is referred to Joannopoulos et al. (1995) and Sakoda (2005).

By the middle of the 20th century, the optics of layered media had been well established, benefiting from the thrust in military requirements during World War II. Improvements were demanded in all areas of optical instrumentation, from binoculars to periscopes, and provided the impetus for industrial activity in this area. The strong modification of light propagation in such systems resulted in a variety of optical properties, ranging from highly reflecting multi-layer coatings to their opposite, the anti-reflective coatings. The reader is referred to details in the classic book by Born and Wolf (1999) for further reading on these topics.

In 1987, the generalization from one-dimensional periodic media (i.e., layered media) to three-dimensional periodic media was independently proposed by Yablonovitch (1987) and John (1987) who also discussed the strong modification of the density of photon states in such systems. Thus, even the spontaneous emission probability for an atom within the photonic crystal, emitting at a frequency in the forbidden band (called the bandgap) was shown to be possibly strongly modulated. Yablonovitch et al. (1991) pursued this work with the demonstration of a face-centered cubic photonic crystal at microwave frequencies. It was demonstrated by calculations (Ho et al. 1990) that a diamond-like lattice structure with a strong enough refractive index contrast could result in a complete bandgap for light propagating in any direction (a three-dimensional bandgap). For the last decade or so, photonic crystals with negligible absorption have become one of the most promising avenues for the development of all-optical circuits. For example, Akahane et al. (2003) have reported optical cavities using two-dimensional photonic crystals with some of the highest ever reported Q-factors ($\sim 10^6$) at optical frequencies.
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An important limitation in controlling the propagation of light in matter came from the fact that the index of refraction could still take positive values only. In fact, a negative refractive index was often seen as being incompatible with the definition of optical density, and hence was often viewed as unphysical. However, careful theoretical considerations showed that a negative refraction could indeed be physical, provided that the medium exhibits other fundamental and necessary properties. The two most important ones were shown to be frequency dispersion (where the permittivity and permeability are not constant with frequency) and dissipation, the two not being independent but related to one another by the necessity of causality. Despite these additional constraints, materials with a negative refractive index had no further reasons to remain hypothetical and the scientific community began a quest for their physical realization.

The germs of the possibility of negative refraction probably first appeared in 1904 during discussions between Sir Arthur Schuster and Sir Horace Lamb regarding the relationship between the group velocity and the phase velocity of waves (see Boardman et al. (2005) for a detailed discussion). The negative group velocity that is possible due to anomalous dispersion at frequencies close to an absorption resonance was the point in contention. For the case of negative refraction, Schuster believed that the group velocity should have a component away from the interface while the phase velocity vector should point inward to the interface. Although Schuster’s conclusion came about from a confusion regarding negative group velocity (the energy flow need not coincide with the group velocity direction in the vicinity of a resonance), it was probably the first consideration of negative phase velocity vectors. In 1944, Mandelshtam considered the possibility of oppositely oriented phase and group velocities (Mandelshtam 1950). He noted that Snell’s law for refraction between two media admitted the mathematical solution of refraction at an angle of \((\pi - \theta_t)\) in addition to the usual angle of refraction at \(\theta_t\), and reconciled it with the fact that the phase velocity still tells nothing about the direction of energy flow. Mandelshtam then also presented examples of negative group velocity structures in spatially periodic dielectric media (Mandelshtam 1945) with the periodicity at wavelength scales. Sivukhin was probably the first to notice the possibility of a medium with negative \(\varepsilon\) and \(\mu\), but rejected it since the possibility of their existence was yet to be clarified.

Viktor G. Veselago first formally considered media with simultaneous negative \(\varepsilon\) and \(\mu\) from a theoretical point of view (Veselago 1968), and concluded that the phase velocity and the energy flow in such media would point in opposite directions. Thus, the media could be considered as having a negative refractive index. He systematically investigated several effects resulting from his conclusions, including the negative refraction at an interface, the negative Doppler shifts, an obtuse angle for Čerenkov radiation, and the possibility of momentum reversal. He also considered the behavior of convex and concave lenses made of such media and also showed that a flat slab of material with \(n = -1\) could image a point source located on one side of the slab onto two
other points, one inside the slab and one on the other side of it (provided that the thickness of the slab was sufficient). His results, however, did not spark much interest at the time and remained an academic curiosity for many subsequent years, primarily because there were no media available at the time which had both \( \varepsilon \) and \( \mu \) negative at a given frequency. The realization of these media had to wait for another 30 years for the development of ideas allowing their experimental realization.

Metamaterials have been the most recent development in this quest for control over light via material parameters, with the recognition that engineered materials, structured in specific manners, can exhibit resonances unique to the structure at certain frequencies. The structures are engineered such that at these frequencies, the wavelength of the electromagnetic radiation is much larger than the structural unit sizes, and thus can excite these resonances while still failing to resolve the details of the structure (shape, size, etc.). Consequently, an array of these structural units (periodic or otherwise) appears to be effectively homogeneous to the radiation and can be well described by effective medium parameters such as a dielectric permittivity \( \varepsilon \) and a magnetic permeability \( \mu \).\(^5\)

### 1.2 The concept of metamaterials

Interestingly, the tremendous interest surrounding media with simultaneously \( \varepsilon < 0 \) and \( \mu < 0 \) arose despite the fact that no natural materials have been, and still are, known to exhibit these properties and all known such media today are artificially structured metamaterials. Although Veselago speculated in his landmark paper (Veselago 1968) that some “gyrotropic substances possessing both plasma and magnetic properties” could be anisotropic examples of left-handed media, to date there is no report of a natural medium with such properties. Therefore, their realization took the path of engineered structures that have been called metamaterials.\(^\dagger\)

The word “meta” implies “beyond” (as in “metaphysics”) and the terminology “metamaterials” today implies composite materials consisting of structural units much smaller that the wavelength of the incident radiation and displaying properties not usually found in natural materials. Although many

\(^5\)It is important to note that the effective medium parameters might have little to do with the bulk material parameters of the medium making up the structures as is discussed in Chapter 3.

\(^\dagger\)The origin of the term Metamaterial has been attributed to R. M. Walser who defined them as “Macroscopic composites having a manmade, three-dimensional, periodic cellular architecture designed to produce an optimized combination, not available in nature, of two or more responses to specific excitation” in 1999 (Walser 2003).
of the ideas of metamaterials have their origin in the theories of homogeniza-
tion of composites (see for example Milton (2002)), metamaterials differ from
those in that they are crucially dependent on resonances for their properties
and the nature of the bulk material of the structural units is often of marginal
importance in determining the effective medium parameters in the relevant
frequency bandwidth. Typically, the resonances in metamaterials can induce
large amounts of dispersion (large changes with frequency) in the effective
medium parameters at frequencies close to resonance. By properly driving
and enhancing these resonances, one can cause the materials parameters \( \varepsilon \)
or \( \mu \) to become negative in a frequency band slightly above the resonance
frequency.

Pendry et al. (1996) first theoretically suggested and later experimentally
demonstrated (Pendry et al. 1998) that a composite medium of periodically
placed thin metallic wires can behave as an effective plasma medium for radia-
tion with wavelength much larger than the spatial periodicity of the structure.
For frequencies lower than a particular (plasma) frequency, the thin wire struc-
ture therefore exhibits a negative permittivity \( \varepsilon \). Although dense wire media
had been considered with much interest as artificial impedance surfaces by
electrical engineers (Brown 1960, Rotman 1962, King et al. 1983), they were
usually considered when the wavelength was comparable to the period of the
lattice and were therefore not really metamaterials per se, for which effective
medium parameters can be defined.

In 1999, Pendry et al. described how one could tailor a medium whose ef-
fective magnetic permeability could display a resonant Lorentz behavior and
therefore achieve negative values of the permeability within a frequency band
above the resonant frequency (Pendry et al. 1999). Again, although simi-
lar structures consisting of loops, helices, spirals or Omega-shaped metallic
particles had been considered earlier by the electrical engineering commu-
nity (Saadoun and Engheta 1992, Lindell et al. 1994) as the basis of artificial
chiral and bianisotropic media, the work reported in Pendry et al. (1999) was
the first to consider them as magnetizable particles that could lead to an
effective negative \( \mu \).

In light of the connection between \((\varepsilon, \mu)\) and the index of refraction \( n \)
expressed in Eq. (1.2), one should immediately wonder what happens to \( n \) when
both \( \varepsilon \) and \( \mu \) are negative. While in usual materials with positive consti-
tutive parameters it is natural to take the positive square root in Eq. (1.2),
\[ n = \sqrt{\varepsilon \mu}, \]
physical and mathematical considerations lead into choosing the
negative square root \( n = -\sqrt{\varepsilon \mu} \) when \( \varepsilon < 0 \) and \( \mu < 0 \). More arguments in
favor of this conclusion are provided subsequently in this chapter and within
the body of this book.

With the basis for a negative permittivity and a negative permeability hav-
ing been laid out, researchers went on to actually experimentally demon-
strate the reality of a negative index medium in a prism experiment at microwave
frequencies (Smith et al. 2000, Shelby et al. 2001b). A photograph of one
of the original metamaterial structures possessing a negative index of refrac-
Figure 1.3 One of the world’s first negative refractive index medium at microwave frequencies reported in Shelby et al. (2001b). The system has negative refractive index for wave propagating in the horizontal plane with the electric field along the vertical direction. The ring-like metallic structures printed on a circuit board provide the negative magnetic permeability while metal wires make the composite acquire a negative dielectric permittivity. (Reproduced with permission from Shelby et al. (2001b). © 2001 by the American Association for the Advancement of Science.)
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Figure 1.4 A capacitor and an inductor form a resonant circuit that can oscillate at \( \omega_0 = 1/\sqrt{LC} \). A capacitor filled with a negative dielectric has negative capacitance, acts as an inductor and can resonate with another usual capacitor. (Reproduced with permission from Ramakrishna (2005). © 2005, Institute of Physics Publishing, U.K.)

Positive permittivity (Raether 1986), whereas materials with negative magnetic permeability are totally novel and can be expected to support the analogous surface plasmon but of a magnetic nature. These surface plasmons on a structured metallic surface can resonantly interact with radiation and give rise to a host of novel electromagnetic effects.

The origin of the surface plasmon can be simply understood as a resonance effect at the interface between two media. Let us consider, for example, the simple case of a capacitor: it is well known that a capacitor can be formed by two parallel conducting plates with an insulating dielectric placed in-between. Filling the gap with a negative dielectric material instead would lead to a capacitor with negative capacitance, which is equivalent to an inductor. Thus two capacitors in a circuit, one filled with a positive dielectric (\( \varepsilon_p \)) and the other filled with a negative dielectric (\( \varepsilon_m \)), can become resonant (see Fig. 1.4). The condition for resonance with two such capacitors turns out to be simply \( \varepsilon_m = -\varepsilon_p \), which is exactly the condition for the excitation of a surface plasmon at the interface between a semi-infinite positive medium and a semi-infinite negative medium in the static limit. Including negative dielectric materials within regular structures of positive dielectrics can therefore yield media in which a variety of resonances can be excited and the structured media would then display many novel phenomena. The excitation of surface plasmons on small implanted metal particles has been exploited for several centuries in Europe to make brilliantly colored glass windows, and it was explained only at the beginning of the 20th century by the Mie theory of light scattering (Bohren and Huffman 1983).

A direct and very novel application of these surface plasmon modes is the perfect lens, which is an imaging device that can preserve subwavelength details in the image and thus overcome the classic diffraction limit (Born and Wolf 1999). It was demonstrated that not only could such a slab of negative refractive medium image a point source in the sense already pointed out in Veselago (1968) for the propagating modes, but that this reconstruction
Figure 1.5 Imaging of an arbitrary object “NANO” by a slab of silver that acts as a super lens. The line width of the “NANO” object is 40 nm. The developed image is found to reproduce subwavelength features of the object to the extent of $\lambda/6$. The figure shows the FIB image of the actual object used at the object plane and the AFM image of the developed image on a photoresist. (Figure kindly supplied by Prof. X. Zhang and based on work published in Fang et al. (2005).)
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Figure 1.6 A two-dimensional transmission line system that displays a negative refractive index. The transmission line has been implemented using lumped circuit elements: essentially it is a microstrip grid loaded with surface-mounted capacitors and an inductor embedded into the substrate at the central node. The figure also shows a probe to detect the near-field radiation. The inset shows the expanded unit cell of the metamaterial. (Reproduced with permission from Iyer et al. (2003). © 2003, Optical Society of America.)

As the field of metamaterials grew rapidly, various communities were drawn into this research field, bringing a variety of viewpoints, expertise, and interesting ideas. This cross-fertilization between so many different fields of physics, mathematics, and engineering is reflected for example in the development of metamaterial antennae (Ziolkowski and Erentok 2006), optical nanonanometers for plasmonics (Muhlschlegel et al. 2005), and a new circuit element approach to the optics or plasmonics of nanosized metallic particles (Alù et al. 2006a). The emerging area of plasmonics quickly became fundamentally related to metamaterials, particularly at optical frequencies. In fact, the very mechanism and designs of negative refractive index media at optical frequencies are, in one way, intimately related to the excitation of these plasmons in the nano-metallic particles making up the structures (Alù and Engheta 2007, Ramakrishna et al. 2007a). Surface plasmon excitations have been shown to be crucial in the mechanisms of several novel optical phenomena such as the extraordinary transmission of light (Ebbesen et al. 1998, Krishnan et al. 2001) through subwavelength-sized hole arrays in metallic films (see Fig. 1.7), large non-linearities due to local field enhancements on rough metal surfaces, single photon tunneling through subwavelength-sized holes (Smolyaninov et al. 2002), etc.
1.3 Modeling the material response

This section reviews some fundamental concepts of continuum electromagnetism that are essential to the ideas of metamaterials. For more in-depth discussions and theoretical details, which are beyond the scope of this book, the reader is referred to standard textbooks of electromagnetic theory such as Landau et al. (1984), Jackson (1999), Kong (2000).

1.3.1 Basic equations

The Maxwell equations are the fundamental equations for the understanding of all electromagnetic and optical phenomena. In their differential form, these
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Equations are written as

\begin{align}
\nabla \cdot \mathbf{E} &= \frac{\rho}{\varepsilon_0}, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t},
\end{align}

where \( \mathbf{E} \) and \( \mathbf{B} \) are the electric field and the magnetic induction, respectively, and \( \rho \) and \( \mathbf{J} \) are the volume charge and current densities, respectively. These equations are called the microscopic Maxwell equations because \( \rho \) and \( \mathbf{J} \) here represent the actual microscopic charge and current densities. In a material medium, for example, \( \rho \) would describe the electronic and nuclear charge distributions. Thus \( \rho \) and \( \mathbf{J} \) would necessarily be complicated and vary extremely fast on very small length scales. Most often, however, we are not interested in the correspondingly fast variations of the electric and magnetic fields over atomic length scales and a macroscopic description is sufficiently accurate. The fundamental Maxwell equations are therefore rewritten at the macroscopic level as

\begin{align}
\nabla \cdot \mathbf{D} &= \rho, \\
\nabla \cdot \mathbf{B} &= 0, \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, \\
\nabla \times \mathbf{H} &= \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t},
\end{align}

where \( \mathbf{E} \) and \( \mathbf{H} \) are the macroscopic electric and magnetic fields, \( \mathbf{D} \) is the displacement field, and \( \mathbf{B} \) is the macroscopic magnetic induction. Similarly, \( \rho \) and \( \mathbf{J} \) are the macroscopic net charge and current densities. Here the microscopic fields are averaged over sufficiently large volumes to yield the macroscopic field quantities wherein the fast variations over small length scales are not observable. Thus, the underlying medium appears homogeneous and shows a homogeneous response to the applied fields. We refer the reader to Jackson (1999) for an insightful derivation of these equations from the microscopic Maxwell equations.

In most materials, the time domain displacement field \( \mathbf{D} \) is directly and linearly proportional to the applied electric field \( \mathbf{E} \), and is a function of the material in which the field propagates. Due to the mass of the electrons

\[ \text{Note that the wavelength of electromagnetic radiation is of the order of } 10^{-2} \text{ m at microwave frequencies and about } 10^{-7} \text{ m for optical (visible) radiation. In addition, the time period of the oscillations are of the order of } 10^{-9} \text{ seconds to } 10^{-15} \text{ seconds, respectively. Therefore, one usually seeks only spatially averaged and time-averaged information, averaged over much longer length scales and time scales.} \]
Introduction

the medium that introduce a certain inertia in the response, $D$ does not vary instantaneously with $E$, but instead is a function of the entire time history of how $E$ excited the medium. A somewhat general form for $D$ can therefore be written in the following form:

$$D(r,t) = \int_{-\infty}^{t} dt' \phi(r; t, t') E(r, t'), \quad (1.5)$$

where $\phi(r; t, t')$ is called the local response function. We assume here that the polarization that sets in a medium depends on the local fields – an assumption that can be violated at small lengthscales due to correlations in the polarization over a given volume of the material. For stationary processes, $\phi(r; t, t') = \phi(r; t - t')$, i.e., all physical quantities depend only on the elapsed time intervals and the above integral becomes a convolution. Frequency domain displacement field and electric field can be defined such as

$$E(r, t) = \int_{-\infty}^{+\infty} d\omega E(r, \omega) e^{-i\omega t}, \quad (1.6a)$$

$$D(r, t) = \int_{-\infty}^{+\infty} d\omega D(r, \omega) e^{-i\omega t}. \quad (1.6b)$$

Introducing these definitions into Eq. (1.5) and using the convolution theorem of Fourier transforms (Arfken 1985), it can immediately be seen that the frequency domains $E(r, \omega)$ and $D(r, \omega)$ are related by the simple linear relation

$$D(r, \omega) = \varepsilon_{0} \varepsilon(r, \omega) E(r, \omega), \quad (1.7)$$

where $\varepsilon(r, \omega)$ is the frequency-dependent dielectric function given by

$$\varepsilon(r, \omega) = \frac{1}{\varepsilon_{0}} \int_{-\infty}^{\infty} d\tau \phi(r; \tau) e^{i\omega \tau}. \quad (1.8)$$

This relation indicates that $\varepsilon$ is dispersive, i.e., function of the frequency $\omega$. The dispersive nature arises from the inertia of the dipoles in a causal medium (due to the mass of the electrons), which defines a material polarization that does not respond instantaneously to the applied fields, but depends on its time history as we have seen. At extremely high frequencies, for example x-rays or $\gamma$-rays, the matter cannot even respond and the “electronic” matter is almost transparent leading to the limit

$$\lim_{\omega \to \infty} \varepsilon(\omega) \to 1.$$ 

We shall see some examples of frequency-dependent dielectric functions in the next section. A similar analysis also holds true for the magnetic permeability $\mu(r, \omega)$, which can be space and frequency dependent.

The expression of $\phi(r, \tau)$ can be obtained from an inverse Fourier transform of Eq. (1.8), and subsequently introduced in Eq. (1.5). Supposing that the orders of integration can be interchanged, it can be shown that the polarization
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is related to the electric field via the Fourier transform of \( \varepsilon(\omega)/\varepsilon_0 - 1 \). The analyticity of this latter function in the upper \( \omega \) plane allows the application of the Cauchy theorem over a contour extending over the real axis, jumping the pole, and closing itself at infinity in the upper plane. This direct complex plane integration provides two relations between the real and imaginary parts of \( \varepsilon(\omega) \), known as the Kramers-Kronig relations, and expressed as (Jackson 1999)

\[
\begin{align*}
\text{Re}(\varepsilon(\omega)) - 1 &= \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} d\omega' \frac{\text{Im}(\varepsilon(\omega'))}{\omega' - \omega}, \\
\text{Im}(\varepsilon(\omega)) &= -\frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} d\omega' \frac{\text{Re}(\varepsilon(\omega')) - 1}{\omega' - \omega},
\end{align*}
\]

where PV denotes the Cauchy principal value. Similar relations hold for the real and imaginary parts of the magnetic permeability \( \mu \). Consequently, in addition to being frequency dispersive, \( \varepsilon \) and \( \mu \) are also required to be complex functions on the account of causality. The imaginary parts account for absorption of radiation in the medium and the total absorbed energy in a volume \( V \) is given by (Landau et al. 1984)

\[
\int_V d^3r \int_{-\infty}^{\infty} \omega \left[ \text{Im}(\varepsilon(\omega))|\mathbf{E}(\mathbf{r},\omega)|^2 + \text{Im}(\mu(\omega))|\mathbf{H}(\mathbf{r},\omega)|^2 \right] \frac{d\omega}{2\pi}.
\]

For example, consider a time harmonic plane wave \( \exp[i(kz - \omega t)] \) propagating along the \( z \)-axis in a dissipative medium with \( \mu = 1 \) and a complex \( \varepsilon \) where \( \text{Im}(\varepsilon) > 0 \). It is clear that the amplitude of the wave decays exponentially due to absorption of the wave as it propagates, which clearly implies that \( \text{Im}(k) > 0 \). This complex wave-vector can be obtained from the Maxwell equations as \( k^2 = \varepsilon\omega^2/c^2 \).

Eqs. (1.9) indicate that the real and imaginary parts of the permittivity (and similarly the permeability) are Hilbert transforms of each other, as illustrated in Fig. 1.8. These relations are derived for material media in thermodynamic equilibrium solely on the grounds of causality. The restriction that they provide on the variation in the real and imaginary parts of material parameters should be regarded as very fundamental. The Kramers-Kronig relations allow an experimentalist, for example, to measure the imaginary part of the permittivity easily by absorption experiments at various frequencies and deduce the real parts of the dielectric permittivity from the imaginary part. An example of this procedure is shown in Fig. 1.8 where the imaginary part of the permittivity is calculated from the real parts by a Hilbert transform with different frequency ranges for the integration. Note that the integrals in the Kramers-Kronig relations involve frequencies all the way up to infinity, whereas it is clear that the effective medium theories break down at high frequencies. However, this does not really affect us in the case of usual optical media since the macroscopic material response functions hold almost down to
Introduction

Figure 1.8 Comparison between the analytic imaginary part of the permittivity (thick black curve on both graphs) and the imaginary part obtained via a Hilbert transform. The analytic expression is 

\[ \varepsilon_r = \frac{1 - (\omega_p^2 - \omega_o^2)}{(\omega_p^2 - \omega_o^2 + i\gamma \omega)} \]

with \( \omega_o = 2\pi \times 10 \text{ GHz} \), \( \omega_p = 2\pi \times 15 \text{ GHz} \), and \( \gamma = \omega_0/2 \). The labels 1, 2, and 3 refer to the subscript \( m \) of \( \omega_m \) and correspond to \( \omega_1 = 2\omega_p \), \( \omega_2 = 4\omega_p \), and \( \omega_3 = 6\omega_p \) respectively.

the level of few atomic distances. Thus, the very high frequency limit is never really probed. In the case of metamaterials, the wavelength is usually larger than the periodicity by only one or two orders of magnitude and this high frequency cutoff, when the homogenization becomes invalid, is easily accessed. Thus these relations should be applied cautiously to metamaterials keeping this in mind: if the effective medium theory itself cannot describe the system, the effective medium parameters obtained from the Kramers-Kronig relations are not meaningful.

1.3.2 Dispersive model for the dielectric permittivity

This section briefly presents a dispersive model for the dielectric permittivity that is due to H. A. Lorentz. The resulting expression is very general and it has been found that many metamaterials exhibit effective constitutive parameters in agreement with this law. As another example, amplifying media such as laser gain media, whose imaginary part of the permittivity is negative, are often modeled by a generalized Lorentz model where the oscillator strength is taken to be negative. The Lorentz model is also a good approximation to the density matrix equations of a weakly perturbed two-level quantum system.
Within this approximation, having a negative oscillator strength corresponds to a population inversion. Note that a similar discussion would hold for magnetizable media and the corresponding magnetic fields. The dispersion of the magnetic permeability in many magnetic materials also exhibits a Lorentz-like dispersion although the resonances usually occur at radio and microwave frequencies. Because of this fundamental importance and relevance to the specific field of metamaterials, we shall introduce the derivation of the Lorentz dispersion law here in order to make it familiar to the reader as well as to bring out its generic features.

The frequency dispersive nature of a medium is related to the polarizability of its basic units, *viz.*, the atoms and molecules. Although one can give an adequate description of dispersion only by a quantum mechanical treatment, a simplified description is possible by using only a few basic results concerning the properties of atoms and molecules. One starts by noting that an applied electric field causes charge separation of the positively charged nuclei and the negatively charged electrons in an atom or molecules. Thus a dipole moment is generated and to a good approximation dominates over the other multipole moments. The induced dipole moments can be determined by the displacements of the charges from their equilibrium positions. The atoms or molecules may additionally have a permanent dipole moment in which case the equilibrium positions of the positive and negative charges do not coincide (we may ignore the motion of the nuclei due to their comparatively large mass). The force on the electrons is given by the Lorentz force:

$$\mathbf{F} = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (1.11)$$

where $\mathbf{v}$ is the velocity of the electrons. We can usually neglect the magnetic field effects as $|\mathbf{B}|/|\mathbf{E}| \sim 1/c$ and most of the speeds involved are non-relativistic.

The electron in an atom or molecule can be assumed to be bound to the equilibrium position through an elastic restoring force. Thus, if $m$ is the mass of the electron, the equation of motion becomes

$$m\ddot{\mathbf{r}} + m\gamma \dot{\mathbf{r}} + m\omega_0^2 \mathbf{r} = -e\mathbf{E}_0 \exp(-i\omega t), \quad (1.12)$$

where $\mathbf{r}$ is the displacement vector, $\omega_0$ is the resonance angular frequency characterizing the harmonic potential trapping the electron to the equilibrium position, and $\omega$ is the angular frequency of the light. Here $m\gamma \dot{\mathbf{r}}$ is a phenomenological damping (viscous) force on the electron due to all inelastic processes. This damping term is extremely important as the oscillating electrons radiate electromagnetic waves and energy, although they can also lose energy in several other manners including collisions.

Using a trial solution $\mathbf{r} = \mathbf{r}_0 \exp(-i\omega t)$, the displacement of the electron is obtained as

$$\mathbf{r}_0 = \frac{-e\mathbf{E}_0/m}{\omega_0^2 - \omega(\omega + i\gamma)}. \quad (1.13)$$
The dipole moment due to each electron is \( p = -e r \) and the polarization, defined as the total dipole moment per unit volume, \( P \), is given by the vectorial sum of all the dipoles in the unit volume. Assuming one dipole per molecule and an average number density of \( N \) molecules per unit volume, one obtains

\[
\mathbf{P} = N \mathbf{p} = \frac{Ne^2 \mathbf{E}}{\omega_0^2 - \omega (\omega + i\gamma)} = \varepsilon_0 \chi_e \mathbf{E},
\]

(1.14)

where \( \chi_e \) is the dielectric susceptibility. Hence one can write for the dielectric permittivity

\[
\varepsilon(\omega) = 1 + \chi_e(\omega) = 1 + \frac{Ne^2/m \varepsilon_0}{\omega_0^2 - \omega (\omega + i\gamma)}. \tag{1.15}
\]

The quantity \( f^2 = Ne^2/m \varepsilon_0 \) is often called the oscillator strength.

Eq. (1.15) is called the Lorentz formula for the dispersion of \( \varepsilon \) whose real and imaginary parts are plotted in Fig. 1.9. The imaginary part, \( \text{Im}(\varepsilon) \), is seen to strongly peak at \( \omega_0 \) and the full width at half maximum is determined by the levels of the dissipation parameter \( \gamma \). The real part, \( \text{Re}(\varepsilon) \), changes in a characteristic manner near \( \omega_0 \) which is consistent with the Kramers-Kronig relations given by Eqs. (1.9).

One should note that the above discussion strictly holds only for a dilute gas of the polarizable objects. In a dense material medium with a much larger concentration, the fields that arise due to nearby polarized objects affect the polarization at any given point. These fields are known as local fields and the polarization that sets in the medium is proportional to the effective field, which is the vectorial sum of the applied field and the local fields.

Needless to say, the actual description of the local fields would be very complicated. On the other hand, in the spirit of homogenization, we can think of each polarizable object to be within a small sphere surrounded by a uniformly polarized medium rather than being a set of discrete dipoles at various locations. Assuming the polarization outside to be a constant, \( P \), one obtains the effective field as**

\[
\mathbf{E}' = \mathbf{E}_{\text{appl}} + \frac{\mathbf{P}}{3\varepsilon_0}.
\]

(1.16)

Thus, we would have to replace the applied electric field in Eq. (1.14) with the effective field. Note that the polarizability \( \alpha \) of the polarizable object is defined by \( p = \varepsilon_0 \alpha E' \), where \( p \) is the induced dipole moment so that the net polarization is expressed as \( P = N \varepsilon_0 \alpha E' \). From Eq. (1.14) we can write

\[
\alpha = \frac{e^2/\varepsilon_0 m}{\omega_0^2 - \omega (\omega + \gamma)}, \tag{1.17}
\]

**One uses the result that the field in a uniformly polarized sphere is \( E = P/3\varepsilon_0 \) in the quasi-static limit.
1.3 Modeling the material response

Figure 1.9  Real and imaginary parts of the dielectric permittivity predicted by the Lorentz model. The parameters for $\varepsilon_1(\omega)$ are $f_1^2 = 0.03\omega_0^2$ and $\gamma_1 = 0.025\omega_0$ and those for $\varepsilon_2(\omega)$ are $f_2^2 = 0.1\omega_0^2$ and $\gamma_1 = 0.01\omega_0$. Note that if the oscillator strength is strong and the dissipation is small enough, the real part of the permittivity can become negative at frequencies just above the resonance frequency as in the case of $\varepsilon_2$.

and the dielectric susceptibility that relates the polarization and the applied fields as

$$\chi_e = \frac{N\alpha}{1 - \frac{N\alpha}{3}}.$$  \hfill (1.18)

The dielectric permittivity thus takes the form

$$\varepsilon = 1 + \chi_e = \frac{1 + \frac{2N\alpha}{3}}{1 - \frac{N\alpha}{3}},$$  \hfill (1.19)

where the local field corrections have been incorporated. This formula is known as the Lorentz-Lorenz formula after the two scientists who came to these conclusions independently and almost simultaneously. For static fields,
an analogous result holds and is known as the Clausius-Mossotti relation for dielectrics.

Finally, we should point out that a crucial approximation made here is that the size of the polarizable objects (atoms and molecules) is very much smaller than the wavelength of radiation. This enabled us to treat all the polarizable objects in the volume as if subjected to the same field with no spatial variation (limit of infinite wavelength). The discussion, however, holds true even for more complicated but small polarizable objects, not just atoms and molecules, which is discussed subsequently.

### 1.4 Phase velocity and group velocity

Shortly after Maxwell introduced the concept of electromagnetic waves, he immediately went about calculating the velocity of these waves and realized that, for a single frequency and in vacuum, they were propagating at the velocity of light (which allowed him to make the connection between the field of electromagnetics and the field of optics). The concept of velocity is fundamental in the study of waves and signals since it provides information on how the wave evolves in space and time, and how fast information can be transferred from one point to another. Yet, one needs to be careful when assigning a physical significance to the various velocities that can be defined.

Let us first take the case of a monochromatic plane wave propagating in the $\hat{z}$ direction. In the time domain, the field is written as $E_y = E_0 \cos(kz - \omega t)$, where $E_0$ is the amplitude of the wave. For a propagating wave, we can track a point of constant phase and realize that it is traveling at a velocity

$$v_p = \frac{dz}{dt} = \frac{\omega}{k}. \quad (1.20)$$

Because of this definition, $v_p$ is called the phase velocity. In the case of free-space, $k = \omega/c$ so that the phase front propagates at the velocity of light. In the case of a more general lossless non-dispersive medium, $k = \omega \sqrt{\varepsilon \mu / c}$ which is a linear function of frequency: the phase velocity is constant, typically the velocity of light in the medium. For yet more general dispersive media, the phase velocity is not a constant with frequency and the phase velocity can be typically larger than the speed of light in the medium. As we shall see subsequently, this does not violate the principle of special relativity since the phase velocity is not associated with a transport of energy, or more strictly, transmission of a signal. Nonetheless, in such a case, various components of a multi-frequency signal propagate at different velocities and cause a phase distortion.

All physical signals are composed of multiple frequencies, i.e., are spread...
1.4 Phase velocity and group velocity

over a certain frequency band. The spectrum of such a wave is never just a Dirac delta function. The assumption of monochromatic plane waves is therefore a theoretical idealization, whereas in the real world, the signal is typically composed of a slowly varying envelope confining a rapidly oscillating wave. The simplest multi-frequency signal is composed of two closely separated frequencies $\omega_0 \pm \Delta \omega$, where $\Delta \omega < \ll \omega_0$, to which correspond the wave-numbers $k \pm \Delta k$. The superposition of the two waves is simply written as

$$E_y = \cos[(k + \Delta k)z - (\omega + \Delta \omega)t] + \cos[(k - \Delta k)z - (\omega - \Delta \omega)t],$$

$$= 2 \cos(\Delta kz - \Delta \omega t) \cos(kz - \omega t).$$

(1.21)

Tracking the constant fronts of the two terms yields two velocities:

1. $kz - \omega t = \text{constant}$ yields the velocity of the rapidly oscillating wave, which is similar to the monochromatic case discussed previously:

$$v_p = \frac{dz}{dt} = \frac{\omega}{k}.$$  

(1.22a)

2. $\Delta kz - \Delta \omega t = \text{constant}$ yields the velocity of the envelope, called the group velocity:

$$v_g = \frac{dz}{dt} = \frac{\Delta \omega}{\Delta k}.$$  

(1.22b)

Intuitively, the group velocity is seen to correspond to the velocity of the envelope or the packet, and corresponds to the velocity of propagation of the energy in many cases.

In the limit of a very narrow-band signal, $\Delta \omega \to 0$ and the group velocity is expressed as

$$v_g = \frac{1}{\partial k/\partial \omega}.$$  

(1.23)

We can also express the group velocity in terms of the phase velocity:

$$\frac{1}{v_g} = \frac{1}{v_p} + \frac{\omega}{\partial \omega} \left( \frac{1}{v_p} \right),$$

(1.24)

which indicates that if there is no frequency dispersion, $v_g = v_p$. In the case of normal dispersion, $\frac{\partial}{\partial \omega}(1/v_p) > 0$ so that $v_g < v_p$. We had mentioned above that $v_p$ can be larger than the velocity of light inside the medium. It can easily be shown that $v_g$ is in fact lower than this limit. Since $v_g$ corresponds to the velocity at which information is carried, it is in compliance with the principle of relativity.

In the case of anomalous dispersion relation, $\frac{\partial}{\partial \omega}(1/v_p) < 0$ so that $v_g > v_p$: the group velocity can be even larger than the speed of light in vacuum. In this case, however, the group velocity loses its meaning as signal velocity, which has to be defined in terms of the electromagnetic energy flow. This issue is discussed in greater detail in Section 6.5.
Finally, let us mention that the definition of the group velocity can be generalized to a vectorial relation as

\[ \mathbf{v}_g = \nabla_k \omega. \]

(1.25)

This gradient relationship indicates that the direction of the group velocity is normal to the iso-frequency contour in the spectral domain. This property is extensively used in Sections 5.1 and 5.2 for example.

1.5 Metamaterials and homogenization procedure

1.5.1 General concepts

One of the crucial ideas in a homogenization procedure is that the wavelength of radiation is several times, preferably several orders of magnitude, larger than the underlying polarizable objects (such as atoms and molecules). In this case, the radiation is sufficiently myopic so as to not resolve the spatially fast varying structural details, but only responds to the macroscopic charge and current densities. Upon averaging in macroscopic measurements, the only remaining important parameters are the frequency-dependent polarization of the individual (atomic or molecular) oscillators driven by the applied fields.

We can apply this idea to a higher class of inhomogeneous materials, such as metamaterials, where the inhomogeneities in a host background are much smaller than the wavelength of radiation, but yet much larger than the “atoms” or “molecules” that the material is composed of. Such a meso-structure would also not be resolved by the incident radiation, and the structure could be driven and polarized or magnetized by applied electromagnetic fields. Particularly near the resonance frequencies (if any), the structures can have a large polarizability. An array of such structural units can then be characterized by macroscopic parameters such as \( \varepsilon \) and \( \mu \) that effectively define its macroscopic response to exciting electromagnetic fields, much like in a homogeneous material.

Metamaterials, in some sense, can be strictly distinguished from other structured photonic materials such as photonic crystals or photonic bandgap materials. In the photonic crystals or bandgap materials the stop-bands or bandgaps arise as a result of multiple Bragg scattering in a periodic array of dielectric scatterers. In fact, the periodicity of the structure in these cases is of the order of the wavelength, and hence homogenization in the classical sense cannot be performed. In metamaterials, the periodicity is by comparison far less important (Chen et al. 2006a), and all the properties mainly depend on the single scatterer resonances. Alternatively, one notes that the small periodicity and small size of the structural units imply that all the corresponding
Bragg scattered waves are evanescent and bound to the single scatterer. Consequently, the properties of a metamaterial are not resulting from interference between waves scattered off different points. Instead, the radiation probes the polarizability of the structural units as it moves through the medium, interacting with the polarizable objects in the same manner as in a homogeneous medium. Note that in the limit of long wavelengths, the phase shifts for the wave across a single structural unit are negligibly small and all units interact with the radiation in a similar manner.

### 1.5.2 Negative effective medium parameters

As discussed in Section 1.3.2, there is a large amount of dispersion in the material parameters at frequencies near the resonance. Below the resonance, the polarization is in phase with the applied driving field, whereas it is $\pi$ out of phase above resonance. If the dissipation is sufficiently small, the resonance can be made very sharp so as to drive the real parts of $\varepsilon$ and $\mu$ even toward negative values when the corresponding driving fields are the electric and the magnetic fields, respectively. Of course, the imaginary parts of $\varepsilon$ and $\mu$ are also large at the resonance frequency and its immediate vicinity.

Thus, negative real parts of the material parameters should be regarded as a natural outcome of an underdamped and overscreened response of a resonant medium. Fundamentally, there is no objection to negative real parts of $\varepsilon(\omega)$ or $\mu(\omega)$ as long as other physical criteria are also satisfied such as causality. The latter implies for example that the frequency dispersive models for the permittivity and the permeability cannot be arbitrary, but should yield constitutive parameters that satisfy the Kramers-Kronig relationship of Eqs. (1.9).

In order to better understand the effect of negative material parameters, consider an isotropic medium where the $\text{Im}[\varepsilon(\omega)] \sim \text{Im}[\mu(\omega)] \simeq 0$, i.e., dissipation is assumed negligibly small at some frequencies (this would typically be a good approximation at frequencies somewhat away from the resonant frequency). We can conveniently characterize most electromagnetic materials by the quadrant where they lie in the complex $(\text{Re}(\varepsilon) - \text{Re}(\mu))$ plane as shown in Fig. 1.10.

**Quadrant 1**: This is the realm of usual optical materials with $\text{Re}(\varepsilon) > 0$ and $\text{Re}(\mu) > 0$. Electromagnetic radiation can propagate through these media and the vectors $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{k}$ form a right-handed triad.

**Quadrant 2**: The usual form of matter that has $\text{Re}(\varepsilon) < 0$ and $\text{Re}(\mu) > 0$ is a plasma of electric charges. It is well known that a plasma screens the interior of a region from electromagnetic radiation. Indeed, all electromagnetic waves are evanescent inside a plasma and no propagating modes are allowed. This is directly expressed by the constitutive relation, which reduces to

$$\mathbf{k} \cdot \mathbf{k} = \varepsilon \mu \omega^2 / c^2 < 0$$  \hspace{1cm} (1.26)

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for a plane wave $\exp[i(k \cdot r - \omega t)]$. Inside such a negative dielectric medium, no real solutions for the wave vector are possible. Note that dielectric materials can also exhibit Lorentz dispersion near an excitonic or optical phonon resonance and $\Re(\varepsilon) < 0$ over a small frequency band above the resonance frequency.

**Quadrant 4:** This quadrant is the dual of quadrant 2, with $\Re(\varepsilon) > 0$ and $\Re(\mu) < 0$. Here, too, a wave incident on a medium of this family decays evanescently within the medium and no propagating modes are sustained. Due to the absence of magnetic monopoles, there can be no exact analogue of an electric plasma but there are natural examples of some antiferromagnetic and ferrimagnetic materials with a resonance at microwave frequencies that exhibit $\Re(\mu) < 0$ in a frequency band above the resonance frequency.

**Quadrant 3:** This is the quadrant of primary interest in this book, directly related to the concept of metamaterials. The properties $\Re(\varepsilon) < 0$ and $\Re(\mu) < 0$ yield a dispersion condition that allows a real wave-vector in the medium, i.e., waves are propagating. Consider the Maxwell equations for a time-harmonic plane wave in the medium:

\[
\begin{align*}
k \times E &= \omega \mu_0 \mu H, \\
k \times H &= -\omega \varepsilon_0 \varepsilon E.
\end{align*}
\]

Since $\varepsilon < 0$ and $\mu < 0$, it is clear that the vectors $E$, $H$, and $k$ form a left-handed triad, which is the property that has given to these media their first very popular (and historical) name of *left-handed media*.$^{11}$ However, the triad of the vectors $E$, $H$, and the Poynting vector $S$ still remains right-handed as $S = E \times H$. This indicates that the Poynting vector and the wave-vector are anti-parallel. This fundamental characteristic indicates that the left-handed media support backward waves, and provides a heuristic argument for declaring that such media have a negative refractive index. As a matter of fact, the refractive index may be defined by $k = \hat{S} n \omega / c$, where $\hat{S}$ is the unit vector along the energy flow. Since $k$ and $\hat{S}$ are in opposite directions, it can be inferred that $n < 0$. A rigorous explanation of this choice of the negative sign for the square root $n = -\sqrt{\varepsilon \mu}$ involves the consideration of causal boundary conditions and dissipative media, whose discussion is postponed to Chapter 5.

### 1.5.2.1 Terminology

We end this chapter by briefly touching upon the terminology used in the literature and in this book for media with $\Re(\varepsilon) < 0$ and $\Re(\mu) < 0$.

Historically, the first name was given in Veselago (1968): *left-handed media* (LHM). As mentioned before, this terminology refers to the left-handed triad formed by the vectors $E$, $H$, and $k$. Some authors, however, criticized this name because of the confusion it may induce with the optical properties of

$^{11}$Note that this terminology is not related to the polarization state of the wave or to the chirality of the medium, but only to the triad of the three vectors $E$, $H$, and $k$. 

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chiral materials, or even with the polarization state of an electromagnetic wave (right-handed or left-handed circularly polarized). In the frame of this book, however, chirality is only briefly mentioned when necessary, and therefore such confusion is very unlikely. We therefore use the historical terminology of left-handed media with consistency and without ambiguity to refer to media that have simultaneously a negative permittivity and a negative permeability.

A second very popular name for these media directly refers the important consequence of negative refraction: negative refractive media (NRM). Although the concept of negative refraction is more reductive than that of a simultaneously negative permittivity and permeability, this terminology has the appeal of referring to one of the most interesting physical properties of these new media so that we shall use it in this book as well.

A few other names have also been proposed in the literature. For example, following Ziolkowski and Heyman (2001), many authors prefer calling them double negative media, as opposed to double positive media when \( \text{Re}(\varepsilon) > 0 \) and \( \text{Re}(\mu) > 0 \), single negative electric media when \( \text{Re}(\varepsilon) < 0 \) and \( \text{Re}(\mu) > 0 \), or single negative magnetic media when \( \text{Re}(\varepsilon) > 0 \) and \( \text{Re}(\mu) < 0 \). Other authors have preferred to call them backward media (Lindell et al. 2001), which describes the negative direction of the phase vector. Another name is negative phase velocity media (NPVM) (McCall et al. 2002), also referring to one of the consequences of a simultaneously negative permittivity and permeability.

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Our standpoint is that no perfect name has yet been found for these media: a name that would encompass all or most of their characteristics and properties, while being simple and evocative enough. Our choice of using either left-handed media or negative refractive media is therefore imperfect, but constitutes a mere terminology choice that had to be made in order to undertake the task of writing of this book.